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THE INITIAL BUSY CYCLE OF A DISCRETE-TIME GI/G/1 QUEUE IN WHICH--ETC(U)
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THE INITIAL BUSY CYCLE OF A DISCRETE-
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INTER-ARRIVAL TIMES ARE NOT
NECESSARILY IDENTICALLY DISTRIBUTED

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THE INITIAL BUSY CYCLE OF A DISCRETE-TIME GI/G/1 QUEUE IN WHICH THE INTER-ARRIVAL TIMES ARE NOT NECESSARILY IDENTICALLY DISTRIBUTED.

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ABSTRACT.

This paper studies a generalization of the discrete-time GI/G/1 queueing system. Here, the inter-arrival times are not necessarily identically distributed. A recursive scheme is derived to obtain the joint distribution of the duration of the initial busy period, the duration of the first idle period and the number of customers served during the initial busy period.

KEY WORDS AND PHRASES: GI/G/1 queue. Inter-arrival times not identically distributed. Initial busy period. First idle period. Number of customers served during the initial busy period.

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The initial busy cycle of a queueing system has always been a subject of interest. It is of great importance in the study of systems in which the cost associated with unproductive idle period is high and thus the server has to be 'switch off' or transfered to another system whenever the original system is empty for the first time. Most of the results obtained in the literature for the initial busy cycle, however, are not very suitable for computer applications. For the GI/G/1 queue, they are presented in terms of Laplace-Stieljes transforms involving complicated operators ([8], equ. 5.18). For the M/G/1 queue, they contain the roots of a certain functional equation within an unit circle ([3], equ. 25). Obtaining numerical information from these results is not an easy task and has never been discussed thoroughly in the literature.

In this paper, we study a discrete-time single-server queueing system in which the inter-arrival times are mutually independent but not necessarily identically distributed; the service times are mutually independent and identically distributed. We shall develope a recursive scheme which enables us to calculate easily the joint distribution of the duration of the initial busy period, the duration of the first idle period and the number of customers served during the initial busy period. This scheme is distribution free; that is, it requires no specific distributional assumptions for the random variables underlying the model. This scheme is suitable

for systems having a very general initial condition. When the system is initiated by the arrival of a customer to an empty queue, this scheme will enable us to obtain information about the normal busy cycle. For the queueing system in which there is a random 'set-up' time for customers who arrive when the server is idle([7],[8]) this scheme is also applicable as the set-up times only affect the initiation of the busy cycle.

For the actual waiting time of each customer in this model, another recursive scheme has been reported in [9].

Let us restrict ourselves to the discrete-time system observed at equally spaced epochs 0, 1, 2, ..., n, ... All events of the system namely arrivals, transfers from queue to service and departures are assumed to occur at instants immediately prior to these epochs. We assume that initially the system is not empty and that there are $\underset{\sim}{k}_0$ ($\underset{\sim}{k}_0 \geq 1$) customers in it at the epoch $n=0$ and the ($\underset{\sim}{k}_0+1$)th, ($\underset{\sim}{k}_0+2$)th, ..., kth, ... customers arrive at the epochs $\underset{\sim}{T}_{k_0+1}, \underset{\sim}{T}_{k_0+2}, \ldots, \underset{\sim}{T}_k, \ldots$ where $0<\underset{\sim}{T}_{k_0+1}<\underset{\sim}{T}_{k_0+2}<\ldots <\underset{\sim}{T}_k<\ldots$ We also assume that the inter-arrival times $\underset{\sim}{t}_k = \underset{\sim}{T}_{k+1} - \underset{\sim}{T}_k$ ($k>\underset{\sim}{k}_0$) are mutually independent but not necessarily identically distributed. We write

$$(1) \qquad a_v^k = \Pr\{\underset{\sim}{t}_k=v\}, \quad v\geq 1, k>\underset{\sim}{k}_0; \quad A_k(z) = \sum_{v=1}^{\infty} a_v^k z^v, \quad z\leq 1, \ k>\underset{\sim}{k}_0.$$

Let the service time of each customer be $\underset{\sim}{S}$. We write

$$(2) \qquad s_j = \Pr\{\underset{\sim}{S}=j\}, \quad j\geq 1; \qquad \Psi(z) = \sum_{j=1}^{\infty} s_j z^j, \quad z\leq 1.$$

The customers are served in order of their arrivals and there is no limit on the waiting room.

First, let $\underset{\sim}{V}_n$ $(n \geqslant 0)$ be the virtual waiting time at epoch n. Let $\underset{\sim}{A}_n$ $(n \geqslant 0)$ be the time difference between n and the time of the first arrival in the interval (n, ∞). Let $\underset{\sim}{C}_n$ $(n \geqslant 0)$ be the total number of customers being in the system at the epoch n=0 or arriving at the system during the interval $(0, n]$. We shall study the multi-variate Markov chain $\{\underset{\sim}{V}_n, \underset{\sim}{A}_n; \underset{\sim}{C}_n\}$.

It is well-known that any stochastic process can be characterized as a Markov process if the full set of random variables needed to specify the state of the process is employed. For the queueing system studied in this paper, the instantaneous state of the process is completely characterized by the bi-variate $\{\underset{\sim}{V}_n, \underset{\sim}{A}_n\}$. In this paper, however, instead of studying the bi-variate $\{\underset{\sim}{V}_n, \underset{\sim}{A}_n\}$, we study the tri-variate $\{\underset{\sim}{V}_n, \underset{\sim}{A}_n; \underset{\sim}{C}_n\}$. We shall show that this will enable us to resolve our problem more fully (See also [6], [7], [9]).

Now let

$$(3) \qquad v^n_{j,\ell;k} = \Pr\{\underset{\sim}{V}_n = j, \underset{\sim}{A}_n = \ell; \underset{\sim}{C}_n = k\} \quad (n \geqslant 0, j \geqslant 1, \ell \geqslant 1, k \geqslant 1).$$

Straightforward arguments result in the following Chapman-Kolmogorov difference equations for $n \geqslant 0$:

$$(4) \qquad v^{n+1}_{0,\ell;k} = v^n_{0,\ell+1;k} + v^n_{1,\ell+1,k} \quad (\ell \geqslant 1, k \geqslant 1) \quad ;$$

$$(5) \qquad v^{n+1}_{j,\ell;1} = v^n_{j+1,\ell+1;1} \quad (j \geqslant 1, \ell \geqslant 1) \quad ;$$

$$(6) \qquad v^{n+1}_{j,\ell;k} = v^n_{j+1,\ell+1;k} + a^{k-j}_{\ell} \sum_{m=1} v^n_{j-m+1,1,k-1} s_m + v^n_{0,1;k-1} s_j a^k_{\ell} \quad (j \geqslant 1, \ell \geqslant 1, k \geqslant 2).$$

Let us now consider a second queueing system, having exactly the same description as the one above but admitting no more customers whenever the system is empty for the first time. It is equivalent to the initial busy period of the original queueing system. The second system will have the same Chapman-Kolmogorov difference equations as (4)-(6) except that the term $v^n_{0,1;k-1}s_j a^k_\ell$ does not appear in (6).

Let the notation for the second queueing system be those for the first system together with the additional symbol $\hat{}$. We say that the chain $\{\hat{\underset{\sim}{V}}_n, \hat{\underset{\sim}{A}}_n; \hat{\underset{\sim}{C}}_n\}$ has $\{0,\ell;k\ \ell\geq 1, k\geq 1\}$ as absorbing states. Cox and Miller [2,p.236] suggest this approach as one suitable method to investigate the distribution of the busy period of the M/G/1 model but do not elaborate any further. (See also [4] and [5]).

From (1), (2), (5)-(6), with the term $v^n_{0,1,k-1}s_j a^k_\ell$ omitted, after some routine calculations, we obtain the following relation between the generating functions

(7) $$\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{n+1}_{j,\ell;k}z^j t^\ell$$

$$= \frac{1}{zt}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{n}_{j,\ell;k}z^j t^\ell - \frac{1}{z}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1,k}z^j$$

$$- \frac{1}{t}\sum_{\ell=2}^{\infty}\hat{v}^{n}_{1,\ell;k}t^\ell - (1-\delta_{k1})S(z)A_k(t)\frac{1}{z}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1,k-1}z^j$$

$$(0<|z|\leq 1,\ 0<|t|\leq 1,\ n\geq 0,\ k\geq 1)$$

where δ_{ij} is the Kronecker delta.

$\hat{v}^n_{1,\ell;k}$ $(n\geq 0, \ell\geq 2, k\geq 1)$ is the probability that the second queueing system becomes empty for the first time after the departure of the k^{th} customer at epoch n+1 and that the (k-1)th customer would have arrived at the system at epoch (n-ℓ) were he allowed to

do so. It is therefore equal to the joint probability that the initial busy period of the original system ends at epoch (n+1), there are k customers served during this period and the duration of the first idle period is $(\ell-1)$. If we denote this by $b^{n+1}_{\ell-1;k}$, we shall have

$$(8)\qquad \xi\sum_{n=0}^{\infty}\sum_{\ell=2}^{\infty}\hat{v}^{n}_{1,\ell;k}t^{\ell}\xi^{n} = t\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}b^{n}_{\ell;k}t^{\ell}\xi^{n} \quad (|t|\leq 1,|\xi|\leq 1,k\geq 1).$$

This relationship indicates that the busy period studied in this paper is what Cohen [1, p. 284] calls the *strong* busy period. A strong busy period continues when an arrival occurs at the instant of departure of the last customer in the system. In other words, two strong busy periods are separated with probability one by an idle period of non-zero duration.

From (7), taking the generating function with respect to time yields

$$(9)\qquad \left[\frac{1}{\xi}-\frac{1}{zt}\right]\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{n}_{j,\ell;k}z^{j}t^{\ell}\xi^{n}$$

$$= \frac{1}{\xi}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{0}_{j,\ell;k}z^{j}t^{\ell} - \frac{1}{z}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1;k}z^{j}\xi^{n}$$

$$- \frac{1}{\xi}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}b^{n}_{\ell;k}t^{\ell}\xi^{n}$$

$$+ (1-\delta_{k1})\Psi(z)\lambda_{k}(t)\frac{1}{z}\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1,k-1}z^{j}\xi^{n}$$

$$(0<|z|\leq 1,\ 0<|t|\leq 1,\ |\xi|\leq 1,\ k\geq 1).$$

If $0<|\xi|\leqslant|t|\leqslant 1$, $k\geqslant 1$, letting $z=\xi/t$ in the above equation we obtain

$$(10)\qquad \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1;k}(\xi/t)^{j}\xi^{n} + \frac{1}{t}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}b^{n}_{\ell;k}t^{\ell}\xi^{n}$$

$$= \frac{1}{t}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{0}_{j,\ell;k}(\xi/t)^{j}t^{\ell}$$

$$+ (1-\delta_{k1})\Psi(\xi/t)\Omega_{k}(t)\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1;k-1}(\xi/t)^{j}\xi^{n}.$$

From (10), denoting by $\beta_{u}^{(v)}$ the coefficient of z^{u} in $\Psi^{v}(z)$ and by $a_{u}^{m,n}$ the coefficient of z^{u} in $\Pi_{v=m}^{n}\Omega_{v}(z)$, by induction, we get

$$(11)\qquad t\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\hat{v}^{n}_{j,1;k}t^{-j}\xi^{j+n} + \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}b^{n}_{\ell;k}t^{\ell}\xi^{n}$$

$$= \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{0}_{j,\ell;k}t^{\ell-j}\xi^{j}$$

$$+ \sum_{v=1}^{k-1}\left[\sum_{u=1}^{\infty}\beta_{u}^{(k-v)}\xi^{u}t^{-u}\right]\left[\sum_{u=1}^{\infty}a_{u}^{v+1,k}t^{u}\right]\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\hat{v}^{0}_{j,\ell;v}t^{\ell-j}\xi^{j}$$

$$- \sum_{v=1}^{k-1}\left[\sum_{u=1}^{\infty}\beta_{u}^{(k-v)}\xi^{u}t^{-u}\right]\left[\sum_{u=1}^{\infty}a_{u}^{v+1,k}t^{u}\right]\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}b^{n}_{\ell;v}t^{\ell}\xi^{n}$$

$$(0<|\xi|\leqslant|t|\leqslant 1,\ k\geqslant 2).$$

Equating the *positive*-degree coefficients of t in the above equation yields

$$(12)\qquad \sum_{n=1}^{\infty}b^{n}_{\ell;k}\xi^{n} = \sum_{j=1}^{\infty}\hat{v}^{0}_{j,\ell+j;k}\xi^{j}$$

$$+ \sum_{v=1}^{k-1}\sum_{u=1}^{\infty}\sum_{w=-j}^{u+\ell}\beta_{u}^{(k-v)}a^{v+1,k}_{u-w+\ell}\sum_{j=1}^{\infty}\hat{v}^{0}_{j,j+w;v}\xi^{j+u}$$

$$-\sum_{v=1}^{k-1}\sum_{u=1}^{\infty}\sum_{w=1}^{u+\ell}\beta_u^{(k-v)}\,a^{v+1,k}_{u-w+\ell}\sum_{m=1}^{\infty}b^m_{w,v}\,\xi^{m+u}$$

$$(0<|\xi|<1,\ \ell\geq 1,\ k\geq 2).$$

Further, equating the coefficients of ξ yields the following relations which will allow us to calculate easily the values of $b^n_{\ell;k}$ $(n\geq 1,\ \ell\geq 1,\ k\geq 2)$ from the *given* initial distribution $\hat{v}^0_{j,\ell;k} = v^0_{j,\ell;k} = \Pr\{V_0=j,\ A_0=\ell;\ C_0=K_0=k\}$:

$$(13)\qquad b^n_{\ell;k} = v^0_{n,\ell+n;k}$$

$$+\sum_{v=1}^{k-1}\sum_{u=1}^{n}\sum_{w=u-n}^{u+\ell}\beta_u^{(k-v)}\,a^{v+1,k}_{u-w+\ell}\,v^0_{n-u,n-u+w;v}$$

$$-\sum_{v=1}^{k-1}\sum_{u=1}^{n}\sum_{w=1}^{u+\ell}\beta_u^{(k-v)}\,a^{v+1,k}_{u-w+\ell}\,b^{n-u}_{w,v}$$

$$(n\geq 1,\ \ell\geq 1,\ k\geq 2).$$

The recurrence is initiated by $b^n_{\ell;1} = v^0_{n,\ell+n;1}$ $(n\geq 1,\ \ell\geq 1)$.

If the initial busy period is initiated by the arrival of one customer to an empty queue, then $v^0_{j,\ell;k} = \delta_{k1}a^1_\ell s_j$ and (13) would enable us to calculate the joint distribution of the duration of a normal busy period, the duration of a normal idle period and the number of customers served during a normal busy period.

For the system in which there is a random set-up time H $(H\geq 0)$ for customers who arrive when the server is idle, we write $h_i = \Pr\{H=i\}$ $(i\geq 0)$ and $v^0_{j,\ell;k} = \delta_{k1}a^1_\ell\sum_{i=0}^{j-1}s_{j-i}h_i$ to obtain information about its normal busy cycle.

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